# *Note on Space Divisions.*

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*The Division of a Plane into Polygons by n unlimited straight lines.*

1. Since the lines are unlimited the finite polygons are *convex;* a division of the plane stretching to infinity between two lines reappears at the opposite side of the plane between the same two lines, and the two infinite divisions make one polygon. Thus all the conclusions given hold after projection, and in obtaining them it is sufficient to consider the simplest form into which the plane can be projected.

2. The total number of polygons is $\frac{1}{2}n(n-1)+1$.* This may be shown by mathematical induction, noticing that an additional line adds one polygon for every segment of it.

3. When $n$ lines form an $n$-gon, the resulting figure may be called an *$n$-wheel.* The $n$-gon may be taken finite and convex; there is a bordering *chain* of $n$ 3-gons; in the exterior angular space between two adjacent sides, there are two of these 3-gons and $n-3$ 4-gons. Thus the distribution of polygons in an $n$-wheel is

$$1\ n\text{-gon},\ \tfrac{1}{2}n(n-3)\ \text{4-gons and}\ n\ \text{3-gons.}$$

4. There is a chain of bordering 3-gons; and the remaining 4-gons are also in chains. Chain $d$ consists of $n$ 4-gons; each 4-gon is bordered on its two interior sides by two 4-gons of chain $d-1$, and angles at their intersection on

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* Cf. Pilgrim; *Zeitschrift für Math. u. Physik;* 24, 188; 1879: "Ueber die Anzahl der Theile, in welche ein Gebiet $k^{\text{ter}}$ Stufe durch $n$ Gebiete $(k-1)^{\text{ter}}$ Stufe getheilt werden kann."

a 4-gon of chain $d - 2$, and its $\left.\begin{matrix}\text{opposite}\\ \text{adjacent}\end{matrix}\right\}$ sides are $\left.\begin{matrix}\text{adjacent}\\ \text{non-adjacent}\end{matrix}\right\}$ in the $n$-gon. If $n$ be even, the outer chain contains $\frac{1}{2}n$ 4-gons.

5. In the distribution with $n$ lines, if there be an $(n - m)$-gon where $n - m > m + 4$, there is only one $(n - m)$-gon and no one of the remaining polygons has more than $m + 4$ sides. This follows by considering first the $(n - m)$-wheel formed by the $n - m$ sides of the $(n - m)$-gon. Thus $n$ lines may give an $n$-wheel, or a distribution with an $(n - 1)$-gon, an $(n - 2)$-gon ... or ($n$ even) an $\frac{n+6}{2}$-gon, ($n$ odd) an $\frac{n+5}{2}$-gon, and there are no other polygons of higher order than 4, 5, 6 ... or ($n$ even) $\frac{1}{2}(n + 2)$, ($n$ odd) $\frac{1}{2}(n + 3)$, respectively.

6. Let $p_r$ denote the number of $r$-gons in the distribution.

$$\sum_{r=3}^{r=n} p_r = \frac{n(n-1)}{2} + 1 \qquad \text{(No. 2)}$$

$$\sum_{r=3}^{r=n} rp_r = 2n(n-1);$$

*i. e.* the number of sides of polygons is double the number of segments of lines. Also,

$$n \text{ even}, \sum_{r=\frac{1}{2}(n+6)}^{r=n} p_r \leqq 1; \; n \text{ odd}, \sum_{r=\frac{1}{2}(n+5)}^{r=n} p_r \leqq 1. \qquad \text{(No. 5)}$$

7. The distribution with one $(n - 1)$-gon is completely specified by the position of the $n^{\text{th}}$ line, *i. e.* by the order in which it crosses the chains of the $(n - 1)$-wheel; for example, a line 121 cuts from a 4-gon of chain 2 a 3-gon vertical to the $(n - 1)$-gon.

The distributions for three simple cases are as follows:

| $(n - 1)$-wheel and | $(n - 1)$-gon. | 5-gons. | 4-gons. | 3-gons. |
|---|---|---|---|---|
| line 121 | 1 | 1 | $\frac{n^2 - 3n - 2}{2}$ | $n$ |
| line 212 | 1 | 2 | $\frac{n^2 - 3n - 6}{2}$ | $n + 1$ |
| line 232 | 1 | 5 | $\frac{n^2 - 3n - 18}{2}$ | $n + 4$ |

8. A table of all possible distributions for $n = 3, \ldots 7$:

| $n$ | | | 7-gon | 6-gons | 5-gons | 4-gons | 3-gons | Total $\frac{n(n-1)}{2} + 1$ | |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 3-wheel | | | | | | 4 | 4 | |
| 4 | 4-wheel | | | | | 3 | 4 | 7 | |
| 5 | 5-wheel | | | | 1 | 5 | 5 | 11 | |
| 6 | 6-wheel | | | 1 | 0 | 9 | 6 | 16 | (No. 3) |
| | 5-wheel and line | 121 | | | 2 | 8 | 6 | | (No. 7) |
| | | 212 | | | 3 | 6 | 7 | | (No. 7) |
| | | 232 | | | 6 | 0 | 10 | | (No. 7) |
| 7 | 7-wheel | | 1 | 0 | 0 | 14 | 7 | 22 | (No. 3) |
| | 6-wheel and line | 121 | | 1 | 1 | 13 | 7 | | (No. 7) |
| | | 212 | | 1 | 2 | 11 | 8 | | (No. 7) |
| | | 232 | | 1 | 5 | 5 | 11 | | (No. 7) |
| | | 323 | | 1 | 3 | 9 | 9 | | |
| | $p_7 = 0$, $p_6 = 0$ | | | | 6 | 6 | 10 | | |
| | | | | | 5 | 8 | 9 | | |
| | | | | | 4 | 10 | 8 | | |
| | | | | | 3 | 12 | 7 | | |

9. Five lines ($n = 5$) always make one pentagon. A conic may be described to touch the five lines; the conic may be projected into an ellipse, and the five tangent lines then evidently form a convex pentagon, which is the pentagon here otherwise noticed.

### *Simple Cases of Division of Space by Planes; and Flat Space of $k$ Dimensions by Flat Spaces of $k - 1$ Dimensions.*

10. Since the dividing flats are unlimited the resulting $k$-fold polyhedroids are convex. A division of the $k$-flat stretching to infinity between $q$ $(k - 1)$-flats reappears in the opposite direction between the same flats, and the two infinite divisions constitute one $k$-fold polyhedroid.

11. Since the $n^{\text{th}}$ dividing $(k-1)$-flat adds one $k$-fold division for every $(k-1)$-fold division of itself, made by its $n-1$ $(k-2)$-flat intersections with the other dividing flats, the following table* of the total numbers of divisions is readily made.

| Divisions of | $n=1$, | 2, | 3, | 4, | 5, | 6, | 7, | 8, | 9, | 10, | 11. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Plane | 1 | 2 | 4 | 7 | 11 | 16 | 22 | 29 | 37 | 46 | 56 |
| Space | 1 | 2 | 4 | 8 | 15 | 26 | 42 | 64 | 93 | 130 | 176 |
| Four-flat | 1 | 2 | 4 | 8 | 16 | 31 | 57 | 99 | 163 | 256 | 386 |

A general expression for the number of divisions in terms of $n$ or of figurate numbers is also readily found. For example, $n$ planes divide space into $\frac{(n-2)(n-1)\,n}{\underline{|3}} + n$ polyhedra, or the $(n-2)^{\text{th}}$ figurate number of the fourth order plus the $n^{\text{th}}$ figurate number of the second order.

12. Four planes divide space into eight tetrahedra, and $n+1$ $(n-1)$-flats divide an $n$-flat into $2^n$ $n$-fold $(n+1)$-hedroids.

The only 3-fold pentahedron has as faces two 3-gons and three 4-gons. The distribution with five planes is ten pentahedra and five tetrahedra. Each tetrahedron has a tetrahedron vertically opposite at each of its four solid angles. This distribution appears from the consideration that ten pentahedra are required to use the twice fifteen 4-gons furnished by the five planes.

There are two 3-fold hexahedra: the A 6-hedron with two 5-gons, two 4-gons and two 3-gons as faces; and the B 6-hedron having six 4-gons as faces. The unique distribution for six planes is obtained from the distribution for five planes upon the passage of a sixth plane, as follows:

| | A 6-hedra | B 6-hedra | 5-hedra | 4-hedra |
|---|---|---|---|---|
| A central pentahedron cut with a 5-gon section gives | 2 | | | |
| Two adjacent tetrahedra " 3-gon " | | | 2 | 2 |
| Two " pentahedra " 3-gon " | 2 | | | 2 |
| One " pentahedron " 3-gon " | | | 2 | |
| One cornering tetrahedron " 4-gon " | | | 2 | |
| Two " pentahedra " 4-gon " | 2 | | 2 | |
| Two " pentahedra " 4-gon " | | 2 | 2 | |
| Uncut | | | 2 | 2 |
| Distribution | 6 | 2 | 12 | 6 |

* Cf. Pilgrim, *loc. cit.*

13. In a 4-flat any four 3-flats meet in a point and divide the 4-flat into eight parts. A fifth 3-flat makes sixteen 4-fold pentahedroids.

There are but two 4-fold hexahedroids; the A $6_4$-hedroid has as faces two 3-fold tetrahedra and four 3-fold pentahedra; the B $6_4$-hedroid has as faces six 3-fold pentahedra.

Six 3-flats divide a 4-flat into six 4-fold pentahedroids, ten B $6_4$-hedroids and fifteen A $6_4$-hedroids, an unique distribution.

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